# HEAT EXCHANGE NEAR THE CRITICAL POINT OF A BODY WITHIN A SUPERSONIC TWO-PHASE JET

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A blunt body located in a supersonic two-phase stream consisting of a gas and large solid particles is heated intensely by collisions and convective heat exchange, which can be several times greater than convective heat exchange in the same flow without particles [1, 2].

Following the theoretical and experimental results of [1-6], the present study will propose relationships permitting calculation of convective thermal fluxes at the critical point and their distribution over the surface of the body over which a supersonic two-phase jet with large solid particles flows. Heat exchange in a supersonic jet with fine particles was considered in [7].

1. We must first consider possible mechanisms for the increase in convective heat exchange between the blunt body and the supersonic two-phase jet. It is interesting to consider the effect on convective heat exchange of the fluid cone formed by particles broken off from the body and the accompanying vortex, departing from the shock layer near the body into the supersonic flow region near the flow critical line. A conical shock wave develops in such a fluid cone [8].

In this case the interaction of shock waves around the body and on the fluid cone can affect convective heat exchange. According to [9], the thermal flux into the body in the region of shock wave collision changes in proportion to the square root of the pressure. At the body critical point the collision of these waves has little effect on heat exchange in light of the small pressure change, which according to the experiments of [8] is even decreased by ~30% as compared to a flow without particles. The effect of the fluid cone upon heat exchange is attenuated still more by the fact that the time of existence of such a cone is significantly less than the duration of the interaction of the two-phase flow with the body, being 10-20% of the latter according to the measurements of [1].

In analogy to experiments on blunt bodies with a needle [10] the fluid cone usually formed near the body critical line should shift the thermal flux maximum from the critical point to the side surface of the body and decrease the thermal flux to the critical point as compared to flow over the body of the same supersonic stream without particles. However, according to the experiments of [1, 2] these effects are not observed. Thus the effect of fluid cones on convective heat exchange can be neglected.

We will propose the following explanation of elevated convective heat exchange in flow of a two-phase stream over a body. Particles or their fragments recoiling up the flow from the body reach the flow shock wave, which as was shown above, intersects a small quantity of particles forming fluid cones. A significantly larger fraction of recoiling particles reaches the shock wave, but because of the abrupt increase in resistance force in the free supersonic jet the wave is only slightly deformed, generating toroidal vortices [8]. These vortices move along gas flow lines to the body in a manner similar to a particle-free supersonic jet turbulent at infinity flowing over a body [3, 6].

For thermal calculations we will use gas parameters at the outer edge of the boundary layer the same as in the absence of particles, since according to [2], the effect of large particles on the gas can be neglected. Gas and particle flow lines at infinity are assumed parallel.

A particle will be termed large if the condition

$$D_p \geqslant 3c_D\rho v\Delta^2/(8\rho_s u_{p\infty}\delta_1), \tag{1.1}$$

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is satisfied, indicating that the particle deviates from its initial direction by not more than the amount $\delta_1$ after tranversing a layer of thickness $\Delta$. Here $\rho$ and v are the gas density and velocity in this layer, perpendicular to the particle velocity $u_{p\infty}$ at infinity; $\delta_1$ is the critical value of the large particle deviation, taken equal to $\Delta$; and $D_p$, $\rho_s$, $c_D$ are the diameter, density, and resistance coefficient of the particle. Inequality (1.1) was obtained from the equation of particle motion for constant values $\rho$, v, $c_D$ and the condition $u_{p\infty} >> v$.

In the shock layer near the axis of flow symmetry, according to [11], we have

$$v/u_\infty = 0.4k\,(\delta + 2\delta_0)/\Delta, \tag{1.2}$$

where the velocity v across the axis is averaged over the distance $\delta$ of particle deviation in a shock layer of thickness $\Delta$; $\delta_0$ is the distance of the particle from the axis of symmetry at infinity; and $u_\infty$ is the gas velocity at infinity; $k = \rho_\infty/\rho$ ($\rho_\infty$, $\rho$ being the gas density ahead of and behind the direct shock wave). With consideration of Eq. (1.2) and the equality $u_{p\infty} = u_\infty$, Eq. (1.1) takes on the form

$$D_p \geqslant 0.15\,(\delta_1 + 2\delta_0)\,c_D\rho_\infty\Delta/(\rho_s\delta_1) \tag{1.3}$$

for the same condition $\delta_1 = \Delta$, while for Eqs. (1.1), (1.3) we can use the relationship $c_D = 1$, valid for a particle moving in a gas with supersonic velocity.

2. In light of the fact that heat exchange in a turbulent boundary layer does not depend upon the mechanism underlying development of disturbances outside the layer which impinge on its outer boundary and support the turbulent regime within, we will make use here of the results of [3, 6] to explain heat exchange between the body and the flow. Those studies considered heat exchange for flow over a body of jet turbulent at infinity.

It was shown theoretically in [3] that the thermal flux to the critical point of a planar ($\nu = 0$) or axisymmetric ($\nu = 1$) body in a perturbed supersonic jet increases with increase in the dimensionless turbulent energy Q, which has the form

$$Q = I^2 Q_1; \tag{2.1}$$

$$Q_1 = \frac{1{,}5}{1+\nu}\,\frac{\rho_0 u_\infty^2}{\mu_0\,(\partial u/\partial l)_0}, \tag{2.2}$$

where $I^2 = (u'/u_\infty)^2$; u' is the intensity of turbulence and velocity pulsation at the outer limit of the boundary layer; $Q_1$ is the dimensionless turbulent energy at $I^2 = 1$; and $\rho_0$, $\mu_0$, $p_0'$, $i_0$, $(\partial u/\partial \ell)_0$ are the density, viscosity, pressure, total enthalpy, and velocity gradient at the body critical point.

We will describe the theoretical dependence proposed in [3] for relative heat exchange $H = \alpha_p^0/\alpha_\ell^0$ at the critical point upon the parameter Q of Eq. (2.1) by the approximate expression

$$H = \begin{cases} 1, & Q \leqslant 275, \\ 0{,}185Q^{0{,}3}, & Q > 275, \end{cases} \tag{2.3}$$

which is close to the approximations used in [4, 5]. Here $\alpha_p^0$, $\alpha_\ell^0$ are convective heat exchange coefficients at the critical point of the body flowed over by a superonic jet without and with particles for one and the same gas parameters at infinity. According to [6], $\alpha_\ell^0$ can be written in the form

$$\alpha_l^0 = 0{,}517\,(1+\nu)^{0{,}5}\,\mathrm{Pr}^{-2/3}\,(\rho_0^*\mu_0^*)^{0{,}5}\,(\partial u/\partial l)_0^{0{,}5},$$
$$\rho_0^* = \rho\,(p_0', i_0^*), \quad \mu_0^* = \mu\,(p_0', i_0^*), \quad i_0^* = 0{,}5\,(i_0 + i_{w0}), \tag{2.4}$$

where $i_{w0}$, $i_0^*$ are the enthalpy at the wall temperature and the effective Eckert enthalpy at the critical point; and $\ell$ is the arc length along the body contour with origin at the critical point; Pr is the Prandtl number. Equation (2.4) coincides well with the expression of Fay and Riddell [12].

TABLE 1

| $Q\cdot 10^{-3}$ | H from [3] | H from Eq. (2.3) |
|---|---|---|
| 0,1 | 1,11 | 1 |
| 0,275 | 1,21 | 1 |
| 0,50 | 1,38 | 1,19 |
| 1 | 1,59 | 1,47 |
| 2 | 1,88 | 1,81 |
| 4 | 2,25 | 2,23 |
| 6 | 2,53 | 2,52 |
| 8 | 2,74 | 2,74 |
| 10 | 2,93 | 2,93 |

A comparison of the results of [3] with calculations by Eq. (2.3) is shown in Table 1, whence it is evident that the accuracy of the approximation increases with increase in the parameter Q.

Note that according to [3] the quantity H depends weakly on the temperature factor $i_{w0}/i_0$: its change over the range $0.1 \leq i_{w0}/i_0 \leq 0.9$ does not exceed 6% with respect to H for $i_{w0}/i_0 = 0.5$, which was used in composing Table 1.

To calculate thermal fluxes with Eqs. (2.1), (2.3) it is necessary to determine the turbulence intensity $I^2$. Expressions were presented in [4, 5] for this quantity, applicable to flow of a supersonic dusty jet over a sphere. However we can take a more general expression for $I^2$, applicable to spheres and end faces, if we use the relationship for the heat exchange coefficient at the critical point from [1], which was obtained to an accuracy of 25% mean square deviation by statistical processing of a large number of experiments involving heat exchange on spheres and end faces flowed over by dusty supersonic jets:

$$H = \alpha_p^0/\alpha_l^0 = \begin{cases} 1, & \chi \leqslant \chi_0, \\ 0{,}098\,(\rho_\infty u_\infty/\alpha_l^0)\,(\mathrm{th}\,\chi)^{0{,}317}, & \chi > \chi_0; \end{cases} \tag{2.5}$$

$$\chi = \rho_{p\infty}u_{p\infty}(1 + G)/(\rho_\infty u_\infty), \quad \chi_0 = (0{,}15k\,\mathrm{Pr}^{2/3}Q_1^{0{,}5})^{-3{,}155}. \tag{2.6}$$

Here $\chi$ is a parameter characterizing the dust concentration in the jet, $\rho_\infty$, $\rho_{p\infty}$ are the gas and particle densities at infinity; $G = G_{er}/(\rho_{p\infty}, u_{p\infty})$, $G_{er}$ are the relative and absolute erosion losses of material per unit body surface near the critical point; and $Q_1$ is calculated with Eq. (2.2). According to [13], the material loss G can be defined in the form

$$G = 0{,}5u_{p\infty}^2/H_{er},$$

where $H_{er}$ is the erosion enthalpy equal to the kinetic energy of particles colliding normally with the body surface required for erosion removal of a unit mass of material from the body surface. The quantity $H_{er}$ can be obtained from experiment (for example, for graphite $H_{er} = 300$ kJ/kg).

Equation (2.5) differs from [1] in the replacement of $\chi$ by tanh $\chi$, which for low jet dustiness ($\chi \ll 1$) does not affect the result ($\tanh\chi \approx \chi$), while for high dustiness it imposes a reasonable limit on the heat exchange coefficient. Roughness was neglected in Eq. (2.5), since its effect on heat exchange does not depart beyond the error range given in Eq. (3.3) below.

From Eqs. (2.3) and (2.5) for turbulence intensity we obtain

$$I^2 = (0{,}34/0{,}75^{1{,}67n})\,k^{3{,}33}\,\mathrm{Pr}^{2{,}22}\,Q_1^{0{,}67}\,(\mathrm{th}\,\chi)^{1{,}06}. \tag{2.7}$$

In deriving Eqs. (2.6), (2.7) the relationships

$$\rho_0^* = \rho_0/0{,}75, \quad \mu_0^* = 0.75^n\mu_0 \tag{2.8}$$

were used with $i_{w0}/i_0 = 0.5$ and gas viscosity in the form $\mu_0 = \mu_0^*(i_0/i_0^*)^n$ (for air n = 0.7, for a constant viscosity gas n = 0).

Equation (2.5) was tested for spheres and end faces by variation of the two-phase jet parameters over the ranges $6 \lesssim M_\infty \lesssim 10$, $4\cdot10^5 \lesssim Re_{\infty1} \lesssim 6\cdot10^7\ m^{-1}$, $750 \lesssim u_{p\infty} \lesssim 1700$ m/sec, $0.05 \lesssim \rho_{p\infty}u_{p\infty} \lesssim 0.5$ kg/(m²·sec) ($Re_{\infty1}$, $M_\infty$ are the Reynolds number at unit length and Mach number for the gas jet at infinity).

The expressions described above make use of velocity gradients at the critical point, which according to [14, 15] for a sphere and end face have the forms

$$(\partial u/\partial l)_0 = \sqrt{2k}\,u_\infty/R_0, \quad (\partial u/\partial l)_0 = 0{,}44k^{0{,}66}u_\infty/R_0 \tag{2.9}$$

($R_0$ is the radius of the sphere or end face).

With consideration of Eqs. (2.1), (2.2), (2.4) and (2.8), Eq. (2.3) transforms in the following manner:

$$\alpha_j^0 = \begin{cases}\alpha_l^0, & Q \leqslant 275,\\ \alpha_p^0, & Q > 275;\end{cases} \tag{2.10}$$

$$\alpha_l^0 = 0{,}517\,(1+\nu)^{0{,}5}\,\mathrm{Pr}^{-2/3}\rho_0^* u_1\,(\mu_0^*\,(\partial u/\partial l)_0/(\rho_0^* u_1^2))^{0{,}5}; \tag{2.11}$$

$$\alpha_p^0 = 0{,}0934\,(I/k)^{0{,}6}\,(1+\nu)^{0{,}2}\,\mathrm{Pr}^{-2/3}\rho_0^* u_1\,(\mu_0^*\,(\partial u/\partial l)_0/(\rho_0^* u_1^2))^{0{,}2}. \tag{2.12}$$

where $u_1$ is the gas velocity immediately behind the incident shock wave; $j = \ell, p$. Equation (2.11) is analogous to Eq. (2.4).

It is interesting to compare Eq. (2.12) with the heat exchange coefficient $\alpha^0$ found in [6] at the so-called turbulent critical point of a body upon which a supersonic turbulent jet impinges:

$$\alpha^0 = c\,(\mu_0^*/\mu_0)^{0{,}6}\,(1+\nu)^{0{,}2}\,\mathrm{Pr}^{-0{,}6}\rho_0^* u_1\,(\mu_0^*\,(\partial u/\partial l)_0/(\rho_0^* u_1^2))^{0{,}2} \tag{2.13}$$

(c is an experimentally determined coefficient). Equation (2.13) was obtained by the integral relationship method using a local heat exchange law. Equations (2.12) and (2.13), derived by other methods, will be equivalent if we neglect the difference between the terms $\mathrm{Pr}^{-2/3}$ and $\mathrm{Pr}^{-0.6}$, and considering Eq. (2.8) write c in the form $c = 0.105(I/k)^{0.6}$.

Thus, convective thermal flux at the body's critical point for low Q values ($Q \leq 275$) are defined by the laminar heat exchange regime of Eq. (2.11), while for high Q ($Q > 275$), the turbulent critical point (TCP) regime of Eq. (2.13) is valid.

It is interesting that according to [6], the heat exchange coefficient along the body directrix for the laminar regime ($j = \ell$) and the TCP regime ($j = p$) obeys a laminar heat exchange law which can be written as

$$\alpha_j/\alpha_j^0 = [2\,(1+\nu)\,\rho_0^*\mu_0^*\,(\partial u/\partial l)_0]^{-0{,}5}\,(\rho^*\mu^* u/x_j)^{0{,}5}, \tag{2.14}$$

$$x_j = \int_0^x L_j dx/L_j, \quad j = l,\ p,$$

$$L_l = \rho^* u\mu^* R^{2\nu}\,(i_l - i_w)^2, \quad L_p = \rho^* u\mu^* R^{2\nu}\,(i_l - i_w)^{1{,}25}\,(i_0 - i_w)^{0{,}75},$$

$$\rho^* = \rho\,(p,\ i_l^*), \quad \mu^* = \mu\,(p,\ i_l^*),$$

$$i_l^* = 0{,}5\,(i + i_w) + 0{,}22\,(i_l - i), \quad i_l = i + 0{,}5\sqrt{\mathrm{Pr}}\,u^2,$$

$$q_l = \alpha_l\,(i_l - i_w), \quad q_p = \alpha_p\,(i_l - i_w)^{0{,}625}\,(i_0 - i_w)^{0{,}375},$$

where x is the coordinate along the body directrix with origin at the critical point; u, i, p are the velocity, enthalpy, and pressure at the outer limit of the boundary layer; $i_\ell^*$ and $i_\ell$ are the defining Eckert enthalpy and the reestablishment enthalpy; R is the body radius, and $q_\ell$, $q_p$ are thermal fluxes.

As is evident from Eq. (2.14) the distributions $\alpha_\ell/\alpha_\ell^0$ and $\alpha_p/\alpha_p^0$ are close to each other, although the quantities $\alpha_\ell^0$ and $\alpha_p^0$ may differ by a factor of several times according to Eqs. (2.11), (2.12).

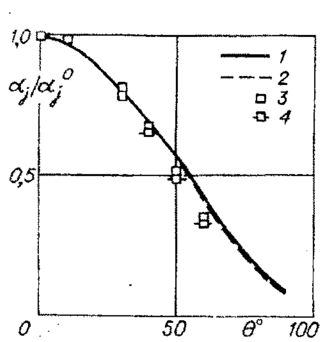


Fig. 1

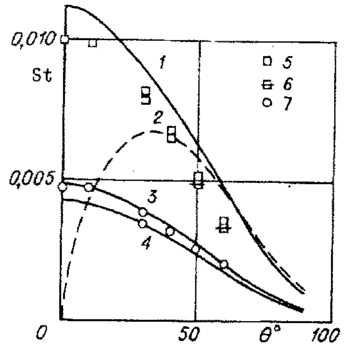


Fig. 2

For comparison with experiment the turbulent heat exchange coefficient $\alpha_t$ will be needed, which according to [6] has the form

$$\alpha_t = 0{,}0296\,\mathrm{Pr}^{-0.6}\,(\rho^* u)^{0.8}\,(\mu^*/x_t)^{0.2}, \quad q_t = \alpha_t\,(i_t - i_w), \tag{2.15}$$

$$x_t = \int_0^x L_t dx/L_t, \quad L_t = \rho^* u\,(\mu^*)^{0.25}\,R^{1.25\nu}\,(i_t - i_w)^{1.25}, \quad i_t = i + 0{,}5\,\mathrm{Pr}^{1/3}u^2,$$

$$\rho^* = \rho\,(p, i_t^*), \quad \mu^* = \mu\,(p, i_t^*), \quad i_t^* = 0{,}5\,(i + i_w) + 0{,}22\,(i_t - i)$$

and is equal to zero at the critical point in contrast to Eq. (2.12) ($q_t$ is the turbulent thermal flux).

3. Calculations with the expressions presented above were compared to the experiments of [2] on relative heat exchange coefficients $\alpha_j/\alpha_j^0$ ($j = \ell$, p) and Stanton number

$$\mathrm{St}_j = \alpha_j/(\rho_\infty u_\infty), \quad j = l, p, t. \tag{3.1}$$

In [2] the thermal flux distribution was measured on a tantalum sphere of diameter D = 76.2 mm over the central angle range $0 \le \theta \le 60°$, within an air two-phase flow with $M_\infty = 1.6$ and $\mathrm{Re}_\infty = 1.88\cdot10^6$, calculated with jet parameters at infinity and dimension D. The jet with braking temperature of 820 K contained silicon carbide ($\rho_s = 3400$ kg/m³) particles 100 μ in diameter with concentration $\chi = 7.3\cdot10^{-4}$ according to Eq. (2.6) at G = 0, so that erosion in the case considered may be neglected. The kinetic energy flux of particles incident on the sphere $q_{pk} = 12.1$ kcal/(m²·sec). These data allow a complete calculation of incident jet parameters.

In evaluating the contribution of particle kinetic energy to the thermal flux [2] did not consider the angle β at which they contacted the sphere surface. A correction to the experimental results of [2] was introduced with the expression

$$\mathrm{St} = \mathrm{St}_{90} + \frac{a_k q_{pk}(1 - \sin^b\beta)}{\rho_\infty u_\infty\,(i_t - i_w)}. \tag{3.2}$$

Here $\mathrm{St}_{90}$ is the experimental Stanton number for β = 90°, $a_k$ is the particle kinetic energy accommodation coefficient, which is taken equal to 0.7 and 0.3, respectively, for erosion-stable and erosion-unstable materials [5] (for tantalum $a_k = 0.7$); b is an exponent equal to 1 or 3; $q_{pk} = 0.5\rho_{pw}u_{pw}^3$ ($\rho_{p\infty}$, $u_{pw}$ are the particle density and velocity near the sphere surface).

The change in particle parameters in the sphere shock layer can be neglected, since according to Eq. (1.3) upon traversal of the shock layer the trajectory of a particle 10 mm distant from the flow axis deviates from its initial direction by an amount $\delta_1 \lesssim 0.01$ mm, while its velocity changes by ∿2% according to the expressions of [13]. Therefore, in Eq. (3.2) we may take $\beta = 90° - \theta$, $q_{pk} = 0.5\rho_{pw}u_{pw}^3$.

The distribution of theoretical relative heat exchange coefficients over the sphere as calculated by Eq. (2.14) is shown in Fig. 1 by curves 1 and 2 for j = p and j = ℓ, which differ only slightly, as was indicated above in considering Eq. (2.14). Those curves agree well with experimental curves 3 and 4, processed with Eq. (3.2) for b = 3 and 1 respectively, with the best agreement occurring for curve 3, i.e., for an inelastic interaction of particle and body, where that portion of their kinetic energy related to the particle velocity component normal to the body is absorbed.

To complete the picture Fig. 2 shows absolute heat exchange coefficients over the sphere, expressed in terms of the Stanton number for the same experimental conditions. Curve 1, characterizing heat exchange in a two-phase jet in terms of $St_p$ and calculated with Eq. (3.1) with consideration of Eqs. (2.14), (2.12), (2.7), (2.6) for G = 0, changes in a manner similar to experimental points 5 and 6, processed with Eq. (3.2) for b = 3 and 1 respectively, differing from them at the critical point, for example, by ~11%. This disagreement is understandable if we consider that the accuracy of approximation (2.5) is ~25%.

For comparison the turbulent Stanton number $St_t$ obtained from Eqs. (3.1), (2.15) is shown in Fig. 2 by line 2, with which curve 1 should coincide after transition of the TCP regime to turbulent far from the critical point. However, the condition for such a transition was not considered in the present study.

The calculation of the laminar Stanton number $St_\ell$ with Eqs. (3.1), (2.14), (2.11) shown by line 4 of Fig. 2 differs from the experimental points 7 due to neglect of the effect of sphere roughness. The increase in heat exchange at the body critical point due to sphere surface roughness can be evaluated with the expressions of [5]

$$\alpha_r^0/\alpha_l^0 = 0{,}2\eta^{0{,}4}, \quad \eta \geqslant 56; \qquad \alpha_r^0/\alpha_l^0 = 1, \quad \eta < 56,$$
$$\eta = (\rho_\infty u_\infty R_{ef}/\mu_0)^{0{,}2}\, h_r/\vartheta_0, \qquad R_{ef} = \sqrt{2k}\,u_\infty/(\partial u/\partial l)_0,$$
$$\vartheta_0 = 0{,}343\sqrt{\frac{\mu_0/\rho_0}{(\partial u/\partial l)_0}}\left(1 - 0{,}286\,\frac{i_{w0}}{i_0}\right), \tag{3.3}$$

where $\alpha_r^0$ is the heat exchange coefficient at the critical point with consideration of roughness; $h_r$ is the mean height of roughness projections on the body surface; $R_{ef}$, $\vartheta_0$ are the effective radius and momentum loss thickness at the body critical point; and $(\partial u/\partial \ell)_0$ is the velocity gradient defined by Eq. (2.9). The results of calculations with Eq. (3.3) agree well with [16] for $\eta \lesssim 10^3$.

The value of $St_r$, calculated with Eqs. (3.1), (2.14) for j = r with consideration of roughness by Eq. (3.3) for $h_r$ = 0.09 mm is shown by curve 3 of Fig. 2, which agrees well with experimental points 7.

It is evident from comparison of curves 1, 3, 4 that the effect of surfce roughness on heat exchange can be neglected in comparison to the effect produced by presence of particles in the jet, as was noted above.

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# GENERAL SOLUTIONS AND REDUCTION OF A SYSTEM OF EQUATIONS OF THE LINEAR THEORY OF ELASTICITY TO DIAGONAL FORM

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Numerous attempts have been made [1-11] to represent stresses or displacements in terms of arbitrary independent functions (for example, harmonic and biharmonic functions) in such a way that the equations of elasticity theory are satisfied identically. We call such representations general solutions. However, up to the present, there has been no single approach to the construction of general solutions. In the present paper we present a method which makes it possible to reduce, in certain cases, a system of differential equations (of linear elasticity theory) with constant coefficients to a simpler system; in particular, to a diagonal system. Moreover, the transformation inverse to the initial system is specified by a transposed or conjugate matrix. Expressions are also obtained for the production of new solutions (operators of symmetry in the sense of group analysis), starting from some concrete solution. The idea of the method is presented briefly in [12]. Explicit formulas are presented for isotropic and transversally isotropic materials, and completeness and generality of the Papkovich–Neiber solution is shown.

The equations of elasticity theory, in the presence of arbitrary anisotropy and the absence of volume forces, have the following form [7] in Cartesian orthogonal coordinates $x_1$, $x_2$, $x_3$:

$$L_{ij}u_j = 0, \quad L_{ij} = L_{ji} = A_{i(kl)j}\partial_{kl} - \rho\delta_{ij}\partial_{\cdot\cdot}, \tag{1}$$

where $u_j$ is the displacement vector; $A_{i(k\ell)j} = (A_{ik\ell j} + A_{i\ell kj})/2$; $A_{ik\ell j}$ is a constant tensor of elastic moduli; $\rho$ is the constant density of the material; $\delta_{ij}$ is the Kronecker symbol; $\partial_k$ indicates differentiation with respect to the coordinate $x_k$; and $\partial_{\cdot}$ indicates differentiation with respect to the time; repeated subscripts indicate summation. Properties of the coefficients $A_{i(k\ell)j}$ were studied in [13-15].

We assume that the matrix L of the operators in relations (1) is similar [16] to some matrix D, i.e., a nondegenerate matrix T exists such that

$$LT = TD. \tag{2}$$

Since $L' = L$ and we assume that $D' = D$, then from Eq. (2) we obtain

$$T'L = DT' \tag{3}$$

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